Odd canonical degree for surfaces of general type

on a joint work with M. Mendes Lopes and R. Pardini

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2^{nd} MaPhyAG Workshop,
July 3^{rd}, 2026,
Universidad Complutense de Madrid (UCM)

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The canonical map

Let S be a smooth complex projective surface.
We consider the birational invariants $p_g(S) = h^0(S, \Omega^2_S)$, $q(S) = h^0(S, \Omega^1_S)$. Let K_S denote a canonical divisor.

Definition

The canonical map is the rational map

$$\varphi := \varphi_{|K_S|} \colon S \dashrightarrow \mathbb{P}^{p_g(S)-1}.$$

We assume throughout that $\Sigma := \varphi(S)$ has dimension 2.
The integer $d := \deg \varphi \geq 1$ is the canonical degree of S.

The basic problem is deceptively simple:

which integers occur as d?

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Even the geography of examples is still surprisingly incomplete.

Beauville's dichotomy

Assume that the canonical image Σ is a surface.

Theorem (Beauville)

There are only two possibilities:

(A) $p_g(\Sigma) = 0$;

(B) $p_g(\Sigma) = p_g(S)$ *and* Σ *is a canonical surface.*

Set $\chi(\mathcal{O}_S) = 1 - q + p_g$. In both cases, if S is a minimal model,

$$9\chi(\mathcal{O}_S) \geq K_S^2 \geq M^2 \geq d \deg \Sigma$$

where M is the moving part of $|K_S|$.

This is the starting point of the classical bounds on d.

A. Beauville, *L'application canonique pour les surfaces de type general*, Invent. Math. 55 (1979).

The Persson–Beauville bound

The inequalities become especially transparent in Beauville's two cases.

Case (A): $p_g(\Sigma) = 0$

$$\deg \Sigma \geq p_g(S) - 2$$

therefore

$$27 - 9q(S) \geq (d-9)(p_g(S) - 2).$$

Case (B): Σ canonical

$$\deg \Sigma \geq 3p_g(S) - 7$$

therefore

$$30 - 9q(S) \geq (d-3)(3p_g(S) - 7).$$

Consequence

The absolute maximum is $d \leq 36$; equality forces

$$p_g = 3, \qquad q = 0, \qquad K_S^2 = 36.$$

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A closer look to the bound

Taking a deeper look to the weaker inequality.

Case (A): $p_g(\Sigma) = 0$

$$\deg \Sigma \geq p_g(S) - 2$$

give

$$d \leq 9\left(1 + \frac{3 - q(S)}{p_g(S) - 2}\right)$$

shows that

- $d \geq 28 \Rightarrow q(S) = 0$
- $d \geq 23 \Rightarrow p_g(S) = 3$ *i.e.* $\Sigma = \mathbb{P}^2$
- $d > 9\left(1 + \frac{2-q(S)}{p_g(S)-1}\right) \Rightarrow \Sigma$ is a surface of minimal degree $p_g(S) - 2$.

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High degrees, recent constructions: odd is rare

Several recent constructions fill previously empty degrees.

d	examples by
1,2,3,4,5,6,7,8,9	several (*see Mendes Lopes-Pardini*)
10,11,14	Fallucca–Gleissner
12,13,15,16,18	Fallucca
16	Persson's classical example;
10,12,14,16,20	Nguyen Bin
16 with q(S)=2, 24	Rito
24 with $q = 1$, 32	Gleissner–Pignatelli–Rito
36	maximal case: Lai-Yeung and Rito

The constructive frontier is now mostly populated by product-quotient surfaces and abelian covers.

C. Gleissner, R. Pignatelli, C. Rito, arXiv:1807.11854; M. Mendes Lopes, R. Pardini,arXiv:2103.01912; B. Nguyen, arXiv:1908.11363, arXiv:2105.02181; F. Fallucca, arXiv:2209.06057.

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When p_g grows: odd is rare

The same inequalities also explain why high canonical degree is a small-p_g phenomenon.

Theorem (Xiao Gang)

For $p_g(S) \gg 0$, if the canonical map is generically finite, then

$$d \leq 8.$$

For p_g large, the possible landscape has a very different shape:

canonical image	large p_g bound	known unbounded examples[1]
$p_g(\Sigma) = 0$	$d \leq 8$	$d = 2, 4, 6, 8$
Σ canonical	$d \leq 3$	$d = 1, 2$

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[1] most unbounded examples with $p_g(\Sigma) = 0$ have $\deg \Sigma = p_g(S) - 2$

Canonical covers of surfaces of minimal degree: odd is rare

A key chapter of the story is formed by canonical covers $\varphi\colon S \dashrightarrow \Sigma$ where Σ is a surface of minimal degree. Interesting are

Mendes Lopes – Pardini International Journal of Math. (1998)

Classification of the canonical maps $S \dashrightarrow \Sigma$ of degree 3 onto a surface of minimal degree with $p_g(S) \geq 4$, assuming the general canonical curve smooth. In this case $p_g \leq 5$, $q(S) = 0$, $K_S^2 \leq 9$ and Σ is a cone.

In contrast with that we have

Gallego – Purnaprajna Trans. AMS (2003-2008)

Classification of the Galois canonical maps $S \to \Sigma$ of degree 4 onto a surface of minimal degree, assuming that $|K_S|$ is base point free. There is an unbounded family.

We constructed recently with Fallucca unbounded families with canonical map of degree 4 such that the limits of the slopes assume countably many different values.

The guiding question: how odd is the odd case?

Let S be a minimal surface of general type and suppose

$$\deg \varphi_{|K_S|} = d > 1, \qquad d \text{ odd}.$$

Question

Are the invariants of such surfaces bounded?

For this problem it is natural to look at the case $\deg \Sigma = p_g - 2$.

Recall that the nondegenerate surfaces $\Sigma \in \mathbb{P}^{p_g - 1}$ with $\deg \Sigma = p_g - 2$ are

1. either the Veronese surface in $\mathbb{P}^5$
2. or a scroll, a smooth surface ruled by lines
3. or the cone over a rational normal curve (also ruled by lines)

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Partial results

A result in this direction on the literature is for the case $|K_S|$ base point free.

Gallego – Purnaprajna 2002

Assume that $|K_S|$ is base point free, and that the canonical degree is odd. Then the canonical image Σ is not a smooth surface ruled by lines.

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Gallego – Purnaprajna 2002

Assume that $|K_S|$ is base point free, and that the canonical degree is odd. Then the canonical image Σ is not a smooth surface ruled by lines.

We can exclude the "isolated" case of the Veronese surface, under the assumption that the general canonical curve is smooth.

Proposition (Mendes Lopes – Pardini – P, arXiv:2604.09216, 2026)

Assume that the general canonical curve is smooth, and that the canonical degree is odd. Then the canonical image Σ is not the Veronese surface in $\mathbb{P}^5$.

The set-up

Let

$$\varphi\colon S \dashrightarrow \Sigma \subset \mathbb{P}^{p_g(S)-1}$$

be the canonical map of a minimal surface of general type with $\dim \Sigma = 2$.

Standing assumptions

1. $d := \deg \varphi$ is **odd**;
2. the general canonical curve $D \in |K_S|$ is smooth;
3. the canonical image Σ is ruled by lines.
4. $p_g(S) \geq 4$;

The hypothesis on smooth canonical curves is crucial.

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Main theorem: the ruled case

Theorem (Mendes Lopes – Pardini – P, arXiv:2604.09216, 2026)

Under the standing assumptions:

1. $p_g(S) \leq d+2$, *and* Σ *is the cone over the rational normal curve of degree* $p_g(S)-2$*;*
2. *if* $p_g(S) = d+2$, *then* $|K_S| = |dC|$ *is base point free,* $C^2 = 1$ *and* $d = 3, 9, 11$*. The case* $d = 3$ *occurs.*

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Idea of the proof of the bound $p_g(S) \leq d+2$:
Let $\beta\colon \hat{S} \to S$ be the blow up of the base points of K_S. By the smoothness of the general canonical curve, $K_{\hat{S}} = |M| + 2E$ with M base point free, E $\beta-$exceptional.
Set $\hat{C} \subset \hat{S}$ for the strict transform of a line. $M\hat{C} = d$ is odd, so $K_{\hat{S}}\hat{C}$ and then $\hat{C}^2$ are odd. So[2] $\hat{C}^2 > 0$. Then by the Index Theorem

$$(d(p_g(S)-2))\hat{C}^2 = M^2\hat{C}^2 \leq (M\hat{C})^2 = d^2 \Rightarrow p_g \leq 2 + \frac{d}{\hat{C}^2} \leq d+2$$

[2] This implies that Σ is cone

A byproduct: a stronger asymptotic bound

Without assuming that Σ is ruled by lines, one still gets new bounds for odd degree.

Corollary

Let d be odd and assume the general canonical curve is smooth. Then

- *if $d = 7$, then $p_g \leq 111$;*
- *if $d = 9$, then $p_g \leq 15$;*
- *if $d = 11$, then $p_g \leq 13$;*
- *if $d \geq 13$, then $p_g \leq 2 + \dfrac{27}{d-9}$.*

In particular, if $p_g \geq 112$, then $d \leq 5$.

The special case $d = 5$

The first unknown odd degree after the triple case is $d = 5$.

Theorem (Mendes Lopes–Pardini–Pignatelli)

Under the standing assumptions, if $d = 5$, then

$$p_g(S) \leq 5.$$

Moreover, if $p_g(S) = 5$, then the induced pencil $|C|$ satisfies

$$C^2 = 1, \qquad K_S C = 5.$$

Let us see, as an example, how we do exclude the case $d = 5$, $p_g = 7$.

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Why the case $d = 5$, $p_g = 7$ do not occur?

If $d = 5$, $p_g = 7$, we know that Σ is the cone over a rational normal curve of degree 5, $|K_S| = |5C|$ is bpf. Here C is the strict transform a line, and $|C|$ is a linear pencil with a simple base point p.

The curves C have genus 4 and p is, as $K_C = 6p$, a Weierstraß point. Standard arguments shows that its sets of gaps is either $\{1, 2, 3, 7\}$ or $\{1, 2, 4, 7\}$. Note that the theta characteristic $3p$ is odd in the first case and even in the second case. By the permanence of the parity of a theta characteristic in families, the set of gaps is the same for all curves C.

Blowing p up we get a fibration $f : \tilde{S} \to \mathbb{P}^1$ with the exceptional divisor E as a section. We have just seen that $h^q(C, kp)$ do not depend on C: then all sheaves $R^q f_* \mathcal{O}(kE)$ are locally free.

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Why the case $d = 5$, $p_g = 7$ do not occur? II

Let us have a better look to the first case, when the set of gaps is $\{1, 2, 3, 7\}$.

Then for $0 \leq k \leq 3$, $rank(f_*\mathcal{O}(kE)) = h^0(f_*\mathcal{O}(kE)) = 1$, so $f_*\mathcal{O}(kE) \cong \mathcal{O}_{\mathbb{P}^1}$.
Pushing forward

$$0 \to \mathcal{O}(3E) \to \mathcal{O}(4E) \to \mathcal{O}_E(4E) \to 0$$

we get

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to f_*\mathcal{O}(4E) \to \mathcal{O}_{\mathbb{P}^1}(-4) \to R^1f_*\mathcal{O}(3E) \to R^1f_*\mathcal{O}(4E) \to 0$$

which forces, as the R^1f_* are locally free of the same rank, $f_*\mathcal{O}(4E) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$ and then $h^0(S, 4C) = 6$. This forces $p_g = h^0(S, 5C) \geq 8$, a contradiction.
The proof of the case $\{1, 2, 4, 7\}$ is similar.

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The hardest exclusion uses an infinitesimal form of the invariance of parity of theta characteristics.

Theorem (Mendes Lopes–Pardini–Pignatelli, 2026)

Let $\Delta \subset \mathbb{C}$ be a disc, and let $\pi \colon X \to \Delta$ be projective and flat, with reduced connected curve fibres C_t, for all $t \in \Delta$.
Let L be a line bundle on X with, setting $L_t := L|_{C_t}$,

$$L_t^{\otimes 2} \cong \omega_{C_t} \qquad \textit{for all } t.$$

Then, for every $k > 0$,

$$h^0(kC_t, L|_{kC_t}) \equiv k\, h^0(C_t, L_t) \pmod 2.$$

Moreover the even integers $kh^0(C_t, L_t) - h^0(kC_t, L|_{kC_t})$ form a non-decreasing sequence.

M. Mendes Lopes, R. Pardini, R. Pignatelli, arXiv:2604.07876.

A useful reformulation

The same infinitesimal parity theorem has a sheaf-theoretic consequence.

Corollary

In the set-up of the theorem, there is a coherent sheaf T on Δ such that the torsion subsheaf of $R^1\pi_*L$ is isomorphic to

$$T \oplus T.$$

This is the form used in the odd-degree paper.

In the critical $d = 5$, $p_g = 5$, $C^2 = 2$ case, a local computation of the torsion of the higher direct image forces a configuration incompatible with this doubled form.

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There are several natural next questions.
In the last theorems, on the infinitesimal permanence of the parity of the theta characteristic, we assume $\dim \Delta = 1$.
J. Kollár asked us if we can extend it to higher dimension as follows:

Remove the assumption on $\dim \Delta$

Let $\pi \colon X \to \Delta$ be projective and flat, with reduced connected curve fibres C_t, $\mathfrak{p}$ a zero-dimensional subscheme of Δ.
Let L be a line bundle on X with $L_t^{\otimes 2} \cong \omega_{C_t}$ for all t. Then, it is true that

$$h^0(\pi^*\mathfrak{p}, L|_{\pi^*\mathfrak{p}}) \equiv (\text{length } \mathfrak{p}) h^0(C_t, L_t) \pmod 2?$$

We proved it for Δ smooth of dimension 1, our proof does not extend to $\dim \Delta > 1$.

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Why does our proof does not extend to $\dim \Delta > 1$?
Our proof follows the idea of Mumford's proof of the fact that the parity of theta characteristic is constant in families. That proof relies, among other things, on the fact that the rank of a skewsymmetric matrix over $\mathbb{C}$ is even. To prove our result we replace that with the following

Lemma

Consider the ring $R = \mathbb{C}[t]/t^k$, $k > 0$. Let M be a skewsymmetric matrix with coefficients in R, let h be the order of M, and consider the map

$$\varphi_M : R^h \to R^h$$

given by multiplication by M. Then the image of φ_M has even dimension as complex vector space.

Open directions

Lemma

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given by multiplication by M. Then the image of φ_M has even dimension as complex vector space.

This does not extend to Artinian rings of higher dimension. If we take $R = \mathbb{C}[x, y]/(x, y)^2$ and

$$M = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{pmatrix}$$

than the image of $\varphi_M : R^3 \to R^3$ has dimension 3 over $\mathbb{C}$.

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Open directions

There are two main natural next questions.

Find or exclude degree $d \geq 5$ examples

Is there any surface S with general canonical curve smooth, $p_g \geq 4$, canonical map of odd degree $d \geq 5$ onto the cone over a rational normal curve?

and

Remove the smoothness assumption

Can one prove any odd-degree boundedness theorem without assuming that the general canonical curve is smooth?

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References for the talk

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Thank you!

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